STOCHASTIC MODEL OF NONIDEAL MIXER. BOUNDARY CONDITIONS IN BATCH SYSTEM*

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Received September 28th, 1982

The earlier proposed general approach for description of the non-ideal mixer is coupled with corresponding boundary conditions for the closed system. Some simplifications in this procedure result in relations which are in agreement with experimental data.

In the last study[1] a unidimensional stochastic model if nonideal mixer has been proposed. On basis of the assumption on random motion of the indicating particle it is possible to describe by this model propagation of scalar quantities (temperature or concentration) in the mixer at turbulent liquid flow. In other studies this general approach has been applied both to the batch[2] and flow[3] mixer. But in both these cases has been considered the so-called „open" mixer *i.e.* additional conditions on its boundaries have not been taken into consideration. In this study an attempt is made to solve some problems which are related with introduction of boundary conditions.

THEORETICAL

GENERAL RELATIONS

Similarly as in the last studies[1-3] the indicating particle is considered, situated in the moment $t = 0$ in the mixer, depicted schematically in Fig. 1. As can be seen from this figure the oriented unidimensional coordinate system with the axis x is chosen where the perpendicular projection of indicating particle motion on this axis is considered. It has been demonstrated[1] that in the case when the effect of molecular diffusion can be neglected it is possible to describe the random motion of the indicating particle in the mixer by use of stochastic differential equations[4]

$$\mathrm{d}V(t) = g[X(t), V(t), t]\,\mathrm{d}t + h[X(t), V(t), t]\,\mathrm{d}W(t) \tag{1}$$

and

$$\mathrm{d}X(t) = V(t)\,\mathrm{d}t\,, \tag{2}$$

* Part LX in the series Studies on Mixing; Part LIX: This Journal *48*, 568 (1983).

where $X(t)$ denotes projection of position of the particle and $V(t)$ is projection of the particle velocity in time t on the axis x and $W(t)$ is the Wiener process. The first right hand side term of Eq. (*1*) is characterising the intensity of non-random, second term of random forces acting on the particle. It has been demonstrated[1,4] that to stochastic differential Eqs (*1*) and (*2*) corresponds the Kolmogorov diffusion equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial}{\partial v}\left[g(x, v, t)f\right] - \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[h^2(x, v, t)f\right] = 0\,, \tag{3}$$

where

$$f \equiv f(x, v, t) = \int_0^{\mathrm{L}}\int_{-\infty}^{+\infty} f'(x, v, t \mid x^0, v^0)\, f^0(x^0, v^0)\, \mathrm{d}v^0\, \mathrm{d}x^0 \tag{4}$$

is probability density of the random process, which means that in the moment t the particle will be located in the space interval $\langle x, x + \mathrm{d}x)$ and that its velocity $V(t)$ will be from the interval of velocities $\langle v, v + \mathrm{d}v)$. The symbol $f'(.)$ denotes transitive probability density characterising the condition that in the moment $t = 0$ the particle is located in the point with the coordinate x^0 and that it has the velocity equal to v^0. Probability density $f^0(.)$ gives the initial distribution of position and velocity of particle and is thus the initial condition of solution of Eq. (*3*). The function $f(.)$ is thus the probability density of the random vector, whose components $X(t)$ and $V(t)$ are defined by Eqs (*1*) and (*2*) and by the initial distribution $f^0(.)$.

Let us discuss the boundary conditions of Eq. (*3*). As concerns classification of partial differential equations relation (*3*) is the equation of parabolic type with two „spacial" coordinates. One of these coordinates v has a physical meaning of velocity and it is possible to consider that the function $f(.)$ is defined for all values v from the interval $(-\infty, +\infty)$ while it is considered that this function and its first derivative converge toward zero at the rise of the absolute value of velocity v beyond all limits,

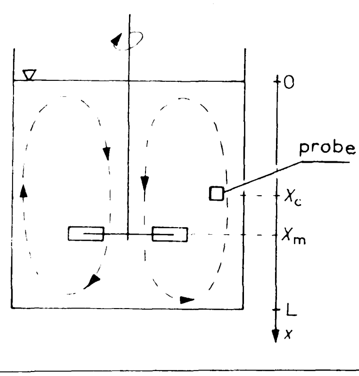

FIG. 1
Motion of fluid particle in the mixed charge

i.e. that there holds

$$\lim_{|v|\to\infty} f(x, v, t) = \lim_{|v|\to\infty} \frac{\partial}{\partial v} f(x, v, t) = 0\,. \tag{5}$$

The "actual" spacial coordinate x is in Eq. (*3*) only a single one, and according to the earlier made considerations, the projection of particle motion on the x axis is studied only (particle motion in the unidimensional space). It is obvious from Fig. 1 that in points $x = 0$ and $x = L$ the function $f(.)$ must in general satisfy certain boundary conditions.

General solution of the problem of boundary conditions is even in the case of unidimensional diffusion very complex; it depends also on the form of coefficients of diffusion equation[5]. In the here considered two dimensional case the situation is even more complex. In the next part it is considered that the boundary of the system in points $x = 0$ and $x = L$ can be considered as regular in the sense of the Feller terminology[5]. In such a case there exist two basic types of "behaviour" of the particle on the boundary of the system: absorption and reflection. The corresponding boundary conditions can be written, for equation of the type (*3*), for the absorbing boundary, by relations[6]

$$\lim_{x\downarrow 0} f(x, v; t) = \lim_{x\uparrow L} f(x, v; t) = 0 \tag{6}$$

and for the reflecting boundary in the form

$$\left.\begin{aligned} \lim_{x\downarrow 0} f(x, v; t) &= \lim_{x\downarrow 0} f(x, -v; t) \\ \lim_{x\uparrow L} f(x, v; t) &= \lim_{x\uparrow L} f(x, -v; t)\,. \end{aligned}\right\} \tag{7}$$

The boundary conditions of the type (*6*) are of significance *e.g.* for mass transfer across the interface. In the next considerations, related to the closed mixer it is always assumed that both boundaries of the system are only refecting and that there does not take place sticking of particles in the walls of the mixer.

There remains to solve the question of relation of boundary conditions (*6*) and (*7*) and random functions $X(t)$ and $V(t)$. Let us write at first the marginal probability density

$$f_x(x; t) = \int_{-\infty}^{+\infty} f(x, v; t)\, \mathrm{d}v \tag{8}$$

which is obviously characterising distribution of position $X(t)$ of the particle at any value of its velocity. Moreover it is possible to define the conditional probability density

$$f_{v|x}(v \mid x; t) = f(x, v; t)/f_x(x; t) \tag{9}$$

which is characterising velocity distribution of particle which is in the moment t located in the point with coordinate x. This „conditional" velocity is thus a function of two arguments and it is formally denoted by the symbol $V(t \mid x)$.

Boundary conditions of the type (*6*) then express the fact that the particle can move in the spacial interval $(0, L)$ only so long till it does not touche some of the boundaries, *i.e.* there holds or

$$X(t) = 0 \quad [t > \tau_0] \tag{10}$$

or

$$X(t) = L \quad [t > \tau_L] \, ,$$

where τ_0, or τ_L is the time of first passage of particle through boundary in the point $x = 0$ or $x = L$.

On the reflecting boundary of the system on the contrary the condition holds

$$\lim_{\substack{x \downarrow 0 \\ t \downarrow \tau}} V(t \mid x) = \lim_{\substack{x \downarrow 0 \\ t \downarrow \tau}} [-V(t \mid x)] \, , \tag{11}$$

or

$$\lim_{\substack{x \uparrow L \\ t \downarrow \tau}} V(t \mid x) = \lim_{\substack{x \uparrow L \\ t \downarrow \tau}} [-V(t \mid x)] \tag{12}$$

which is expressing the fact that the particle is on the corresponding considered boundary in the moment τ elastically reflected and returns back into the interval $(0, L)$.

The general unidimensional model with so defined boundary conditions will be now applied to the simpler concrete case for which it is possible to find the explicite resulting relations.

MODEL OF PARTICLE MOTION

In the recent studies[2,3] it has been demonstrated that useful results – very often they can be solved even by analytical methods – might be obtained at relatively simple assumptions. It has been considered that forces which act on the fluid particle and which are described by coefficients $g(.)$ and $h(.)$ in Eq. (*1*) are at most linear functions of variables x and v. Similar procedure is used also in this case.

Similarly as in the earlier study[2] is here considered a homogeneous liquid situated in a cylindrical vessel (Fig. 1). The x axis of the earlier selected unidimensional coordinate system is identical*) with the axis of cylindrical symmetry of the vessel and as its

* In Fig. 1 the axis x is drawn outside the mixer for clearness.

origin is chosen the point in which the liquid surface is intersected by this axis. In the axis of cylindrical symmetry of the vessel is located a rotary impeller which is causing an axial flow of the batch so that the plane of horizontal symmetry of the impeller rotor is cutting the axis x in the point x_m. In the moment $t = 0$ a solution of negligible volume containing the indicating compound is injected on the surface of the stationary mixed liquid. Particles – molecules – of the indicating compound have the same density as the liquid and move randomly inside the vessel.

As has been already stated an attempt is made here to describe only the vertical component of particle motion while other simplifications are made: it is assumed that after injection of the indicating compound on the surface of the batch in the vessel the particles will move mostly at first in the descending stream and after reaching the bottom of the vessel they will be situated in the ascending stream. After reaching the surface the particles will be again situated in the descending stream. The described procedure is continuously repeated.

Further assumptions concerning the forces acting on the particle in the mixed charge are introduced:

1) By mutual action of the rotating impeller and walls of the vessel a nonrandom force originates, whose magnitude is constant in the whole volume of the charge and differs only as concerns the direction of the descending and ascending streams of the batch;

2) nonrandom friction force directly proportional to particle velocity and oriented against the direction of its motion;

3) Random force proportional to the Wiener process. Proportionality coefficients are constant for the given mixing conditions. On basis of these assumptions it is possible to write Eq. (*1*) in the form

$$\mathrm{d}V(t) = [\pm b - \alpha V(t)]\,\mathrm{d}t + c\,\mathrm{d}W(t)\,. \qquad (13)$$

This equation does not include $X(t)$ and it is thus possible to find independently the distribution function or probability density $f_v(v;t)$ of velocity of the indicating particle. The Kolmogorov equation[1] holds

$$\frac{\partial f_v^{\pm}}{\partial t} + \alpha \frac{\partial}{\partial v}[(\pm\beta - v) f_v^{\pm}] - \alpha\,\varepsilon^2 \frac{\partial^2 f_v^{\pm}}{\partial v^2} = 0\,, \qquad (14)$$

where $\beta = b/\alpha$ and $\varepsilon = c/\sqrt{(2\alpha)}$. Relations (*13*), (*14*) and others are formal registration of always two independent equations in which is always the upper index related to the descending and lower to ascending streams of the charge.

As has been already stated, liquid motion in the mixer is quasistationary and the initial volume of the solution of the indicating compound is negligible in comparison to the volume of the charge. It is thus possible to assume with sufficient accuracy that the velocity of particles will be from the beginning of the operation a stationary random function with the probability density

$$f_v^{\pm}(v; t) = f_v^{\pm}(v) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\{-[(v \pm \beta)^2/2\varepsilon^2]\} . \qquad (15)$$

Here β is the expected mean velocity of indicating particles and ε^2 variance of their velocities around this mean value.

With regard to Eqs (*1*) to (*3*) and (*14*) it is finally possible to write the Kolmogorov's equation for the probability density characterising distribution of the vertical projection of position and of particle velocity in the descending and ascending liquid streams by relation

$$\frac{\partial f^{\pm}}{\partial t} + v\frac{\partial f^{\pm}}{\partial x} + \alpha\frac{\partial}{\partial v}[(\pm\beta - v)f^{\pm}] - \alpha\varepsilon^2\frac{\partial^2 f^{\pm}}{\partial v^2} = 0 . \qquad (16)$$

For this differential equation the initial conditions holds

$$f^{0\pm}(x^0, v^0) = \delta(x^0) f_v^{\pm}(v^0) \qquad (17)$$

which is expressing the fact that in the initial moment all indicating particles are on the surface of the liquid and distribution of their velocities is given by Eq. (*15*).

It is possible to demonstrate that particular solution of Eqs (*16*) with the given initial condition is the two-dimensional normal distribution in the form (see Appendix)

$$f_n^{\pm}(x, v; t) = (1/2\pi\,[h_{xx}h_{vv} - h_{xv}^2]^{1/2}) . \qquad (18)$$

$$. \exp\{-[h_{vv}(x - x_n^{\pm})^2 - 2h_{xv}(x - x_n^{\pm})(v - v^{\pm}) + h_{xx}(v - v^{\pm})^2]/2(h_{xx}h_{vv} - h_{xv}^2)\} ,$$

whose parameters are given by relations

$$\begin{aligned} x_n^{\pm} &= \pm\beta t \mp 2nL , \qquad (19)\\ v^{\pm} &= \pm\beta ,\\ h_{vv} &= \varepsilon^2 ,\\ h_{xv} &= \varepsilon^2[1 - \exp(-\alpha t)]/\alpha ,\\ h_{xx} &= 2\varepsilon^2[\alpha t - 1 + \exp(-\alpha t)]/\alpha^2 . \end{aligned}$$

Symbol n is an arbitrary integer and gives the number of reflections of the indicating particle on boundaries of the closed interval $\langle 0, L \rangle$. The quantity n can become also negative which denotes the fact that individual particles can even move "against" the direction of flow of the liquid stream.

Let us define the "overall" probability density $f(x, v; t)$ so that the symbol $f(x, v; t)$ . . $dx\, dv$ denotes such probability that the particle is located within the interval $\langle x, x + dx \rangle$ either in the descending or ascending stream of the charge and that it has a velocity from the interval $\langle v, v + dv \rangle$ so that the relation holds

$$f(x, v; t) = \sum_{n=-\infty}^{+\infty} [f_n^+(x, v; t) + f_n^-(x, v; t)] . \qquad (20)$$

It is necessary to realise that only sums $\sum f_n^+$ and $\sum f_n^-$ are the solution of each of equations (*16*) considered individually; the alone function $f(x, v; t)$ does not satisfy these equations. But it is easy to prove that it suits the boundary conditions (7) on both ends of the interval, *i.e.* that it is expressing the fact that in the moment of reflection the particle is situated practically simultaneously both in the ascending and descending streams of the charge. More accurately this concerns convergence of the just discussed summations from left and right on the time axis to this moment as is indicated in Eqs (*11*). It is obvious that the sums $\sum f_n^+$ and $\sum f_n^-$ then describe the situation separately in the descending and ascending streams of the charge. So is the problem in general form solved.

Earlier[1] it has been demonstrated that the marginal probability density $f_x(.)$ defined by relation (*8*) is proportional to the expected value of concentration of indicating particles in unidimensional space. Proportionality coefficient is for simplicity considered as equal to one and after integration of Eq. (*20*) indicated in Eq. (*8*) the relation is obtained

$$E[Q(x, t)] = \sum_{n=-\infty}^{+\infty} 1/\sqrt{(2\pi h_{xx})} \exp[-(x - x_n^+)^2/2h_{xx}] + $$
$$+ \sum_{n=-\infty}^{+\infty} 1/\sqrt{(2\pi h_{xx})} \exp[-(x - x_n^-)^2/2h_{xx}] . \qquad (21)$$

This function is on basis of the earlier made considerations[1] expressing the mean value of concentration of indicating particles in differential volume of the charge between horizontal planes which intersect coordinate axis in points x and $x + dx$. Mean concentration in the descending stream is then expressed by first sum, in the ascending by the second sum.

It is known that these sums converge fast for small values of variance h_{xx}. For large values it is better to use the equivalent relation[7]

$$E[Q(x, t)] = (1/L)[1 + 2\sum_{n=1}^{\infty} \exp(-n^2\pi^2 h_{xx}/L^2) \cos(n\, \pi x/L) \cos(n\, \pi\beta t/L)] . \qquad (22)$$

From it with regard to the last one of the system of Eqs (*19*) it is immediately obvious that in very long time after adding of indicating particles their perfect homogenisation in the whole volume of the charge takes place.

In the same study[1] has been defined also the relation for variance of concentration of indicating particles which will be looked for as the sum of variances in the descending and ascending streams of the charge

$$\operatorname{Var}[Q(x,t)] = E[Q^2(x,t)] - E^2[Q(x,t)] =$$
$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f^{+}_{nx|v} f^{+}_{mx|v} f^{+}_{v}\, dv - \int_{-\infty}^{+\infty} f^{+}_{nx|v} f^{+}_{v}\, dv \int_{-\infty}^{+\infty} f^{+}_{mx|v} f^{+}_{v}\, dv\right] +$$
$$+ \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f^{-}_{nx|v} f^{-}_{mx|v} f^{-}_{v}\, dv - \int_{-\infty}^{+\infty} f^{-}_{nx|v} f^{-}_{v}\, dv \int_{-\infty}^{+\infty} f^{-}_{mv|v} f^{-}_{v}\, dv\right], \quad (23)$$

where functions $f^{\pm}_{v}(.)$ are defined by relations (*15*) and the conditional probability density $f_{nx \cdot v}$ is given by relation

$$f^{\pm}_{nx|v}(x \mid v; t) = f^{\pm}_{n}(x, v; t)/f^{\pm}_{v}(v)\,. \quad (24)$$

After substitution from relations (*15*), (*18*) and (*24*) into Eq. (*23*) and after integration an explicit relation for variance of concentration of indicating particles is obtained in the form

$$\operatorname{Var}[Q(x,t)] = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left\{\frac{1}{2\,\pi h_{xx} \sqrt{(1-r^4)}}\,\cdot\right.$$
$$\cdot \exp\left[-\frac{(x-x^{+}_{n})^2 + (x-x^{+}_{m})^2 - 2r^2(x-x^{+}_{n})(x-x^{+}_{m})}{2h_{xx}(1-r^4)}\right] -$$
$$\left. - \frac{1}{2\pi h_{xx}} \exp\left[-\frac{(x-x^{+}_{n})^2 + (x-x^{+}_{m})^2}{2h_{xx}}\right]\right\} + \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \{\ldots\}, \quad (25)$$

where the symbol in the second brace denotes the same term as in the first one, with the difference that symbols x^{+}_{n} and x^{+}_{m} are interchanged by symbols with negative sign. The symbol r denotes correlation coefficient between the position and velocity of particle and is defined by relation

$$r = h_{xv}/(h_{xx}h_{vv})^{1/2} = (1 - \exp(-\alpha t))/\sqrt{2}[\alpha t - 1 + \exp(-\alpha t)]^{1/2} \quad (26)$$

with regard to the last three equations of the system (*19*).

From Eqs (*25*) and (*26*) results that for large times t converges the fourth power of the correlation coefficient faster than the variance h_{xx}. The value of the first term in the brace of Eq. (*25*) then converges to the value of the second term so that there results that the variance of concentrations converge to zero. This fact is strengthening the remark made behind Eq. (*22*) in the respect that after sufficiently long time concentration of indicating particles in the charge will reach a constant value. Eqs (*21*) or (*22*) and (*25*) thus qualitatively correctly describe homogenisation of indicating compound in the mixed charge with time. Thus an attempt has been made for their experimental verification.

EXPERIMENTAL

Experiments were performed in the unit described earlier[2]. Electrolytic conductivity of the charge has been measured after adding about 3 ml of concentrated solution of sodium chloride on its surface. The measuring prove was formed (Fig. 2) by four platinum wires stretched on the skeleton made from isolating material so that they formed vertical edges of the hypothetic cube with volume about 4 cm^3. This probe has been situated in the half of vertical height of the charge and on the radius which was cutting in half the angle between two neighbouring radial baffles. Radial distance of the center probe from the vessel axis has been equal to 2/3 of its radius. This position of the pro-

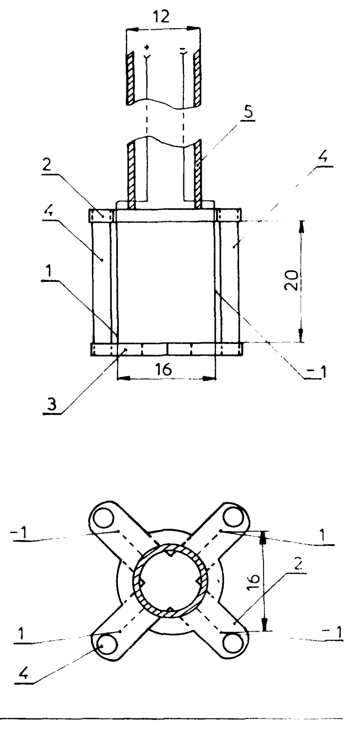

FIG. 2

Probe for measurement of electrolytic conductivity of solution **1** Pt wires connected with the positive pole of power supply, —*1* Pt wires connected with negative pole of power supply, **2** upper arm of the probe, **3** lower arm of the probe, **4** spacer, **5** pipe for fixing the probe and insulation of wires be should have enabled simultaneous measurement of concentration in the ascending and descending streams of the charge.

A six blade turbine with ratio of impeller diameter to the vessel diameter equal to 1/4 and 1/3 has been used. Distance of the impeller rotor from the vessel bottom was equal to 1/3 of the charge height. Rotational speed of the impeller has been varied within the range from 1·67 to 8·33 s^{-1}.

Similarly as in the last study[2] always 20 parallel measurements under the same experimental conditions were performed. From so obtained experimental data the mean value and variance of dimensionless concentration as time sequences were calculated. Dimensionless concentration has been given by relation

$$Z(x_c, t) = Q(x_c, t)/Q(x_c, \infty)\,, \tag{27}$$

while as $Q(x, \infty)$ has been chosen the final concentration *i.e.* such value which in continuing experiment remained constant. The vertical coordinate of the probe center x_c (Fig. 1) was equal to $L/2$. Mean value and variance of dimensionless concentration $Z(x_c, t)$ in time t were calculated according to relations

$$\left.\begin{aligned} \bar{Z}(L/2, t) &= \frac{1}{n}\sum_{i=1}^{n} Z_i(L/2, t)\,, \\ S^2(L/2, t) &= \frac{1}{n-1}\sum_{i=1}^{n}\left[Z_i^2(L/2, t) - \bar{Z}^2(L/2, t)\right] \end{aligned}\right\}, \tag{28}$$

index i denotes No of experiment in series of n measurements under identical conditions. Quantities $\bar{Z}(L/2, t)$ were used for determination of parametes of the model.

Before calculation of parameters of Eqs (*21*), (*22*) and (*25*) these relations were rearranged into dimensionless form; Eqs (*21*) and (*22*) were multiplied by a constant L, Eq. (*25*) was multiplied by L^2. Dimensionless parameters ξ, ϱ and γ and dimensionless time θ were introduced

$$\begin{aligned} &\xi = x/L\,, \quad \varrho = \beta/\alpha L\,, \quad \gamma = \varepsilon^2/\alpha^2 L^2\,, \\ &\theta = \alpha t\,. \end{aligned} \tag{29}$$

From so arranged relations then the values of parameters α, ϱ and γ were determined by the method of nonlinear regression both from experimentally determined time dependences of mean dimensionless concentration and from variance of this concentration. Parameter ξ has allways reached a constant value $\xi = 0{\cdot}5$ (corresponds to location of the probe in the half of the charge-height, see Fig. 1). For calculation the Marquart[8] optimisation procedure was used. The needed partial derivatives of dimensionless forms of Eqs (*21*) and (*22*) were determined numerically.

With regard to the fact that in calculations by use of Eqs (*21*), (*22*) and (*25*) it is necessary to calculate sums of infinite series we have studied practically the rate of convergence of these sums to limiting values. The values of parameters α, ϱ and γ were chosen in a wide range (for $\xi = 0{\cdot}5$) and for different number of terms of series their sums were evaluated. It has been found that convergence of these partial sums is very fast (it has not been necessary to use in calculation the alternative relation (*22*)); practically suffices to obtain a sufficiently accurate sum of 10 to 20 terms of the series both for positive and negative values of the summation index. At regression calculation of estimates of parameters α, ϱ and γ were chosen the summation limits so that the required accuracy of computation is met.

RESULTS

Experimental results (time series) were arranged into sets, whose individual terms differ only by rotational speed of impeller. Initial estimates of parameters α, ϱ and γ necessary for nonlinear regression were determined by trial and error for the first term of the set and for other terms as initial estimates were taken results of preceeding calculation. The results of calculations were thus values of parameters α, ϱ and γ in dependence on the rotational speed of the impeller at the given configuration of the mixed system.

It results from theoretical part and discussion of this paper that the value of parameter α should be independent of rotational speed of the impeller. Estimates of parameter α obtained by the above described calculations practically corresponded to this conclusion but their values fluctuated randomly around some constant mean value. Deviations were obviously due to strong correlation among pairs of parameters $\alpha - \varrho$ and $\alpha - \gamma$, thus the calculation of parameter estimates ϱ and γ were repeated with the constant value of parameter α (equal to the given mean value). So were obtained the corrected values of parameters ϱ and γ while their dependence on rotational speed of impeller is given in Fig. 3. By comparison of sums of square deviations for both calculation variants it has been found that the experimental and calculated dimensionless concentrations of indicating particles are practically identical in both these cases.

In Fig. 4 the example is given of time dependence of dimensionless concentration $\bar{Z}(L/2, t)$ where are the experimentally determined values compared with results of both these variants of calculation. In Fig. 5 is demonstrated the time dependence of variance of this concentration calculated according to Eq. (*25*), where for the calculation of variance were used parameters α, ϱ and γ calculated from the change in concentration according to the second method of calculation.

DISCUSSION

By the earlier described procedure[2] it has been tested how suitably is the model describing changes in the infinitely long spacial interval and it has been found that with a sufficient accuracy (according to criteria quoted in the mentioned study[2]) it describes experimental data. Comparison of sum of square of deviations in both these cases enables to judge that the three parameter model, derived in this study (the fourth parameter was in all studied experimental series a constant) describes the experimentally obtained concentration dependence better than the earlier derived four parameter model[2]. A good agreement of experiments with theoretical concentration dependence can be also seen from the example given in Fig. 4.

Model derived in this study enables even (in some cases at least qualitatively) to express the dependence of variance of concentration on time as in obvious from

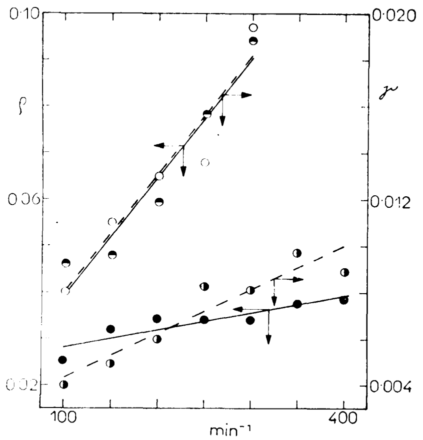

FIG. 3

Dependence of parameters ϱ and γ on rotational speed of impeller. ——— ○ ϱ, ----- ◓ γ turbine $d = 100$ mm, vessel with heating coil, $\alpha = 0{\cdot}8$; ——— ● ϱ, ----- ◑ γ turbine $d = 76$ mm, $\alpha = 1{\cdot}0$

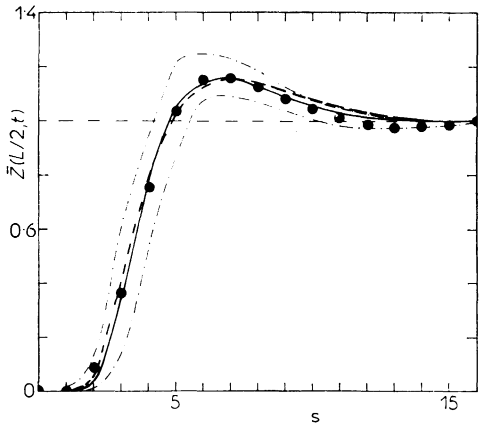

FIG. 4

Experimentally determined mean dimensionless concentration in comparison with the calculated one by use of the model. ● experimental values, ——— variable α, ----- constant α, ·-·-·-· limits of 95% reliability interval

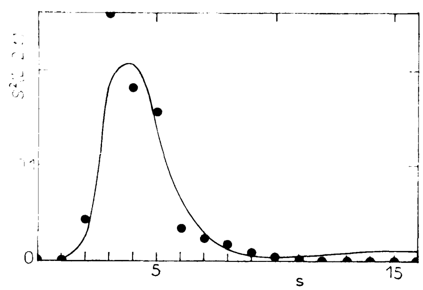

FIG. 5

Experimentally determined variance of concentration in comparison with the one calculated according to the model. ● experimental values, ——— dependence according to the model Eq. (*23*) into which were substituted parameters calculated by nonlinear regression from Eq. (*21*)

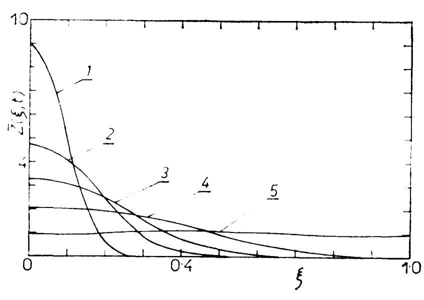

FIG. 6

Dependence between dimensionless concentration of indicating compound and dimensionless linear coordinate in dependence on time interval t from the moment of injection of the indicating compound on surface of the charge; *1* $t = 1$s, *2* $t = 2$s, *3* $t = 3$s, *4* $t = 5$s, *5* $t = 10$s

Fig. 5. The agreement of experimental data and theoretical dependence can be considered in this case as very good when it is realised that for their calculation were used parameters calculated by use of relation (*21*). This fact also confirms that the proposed model is suitable.

As has been already given in the theoretical part, the model has described correctly equality of concentrations of the indicating compound in time and space of the batch. This fact is demonstrated in Fig. 6, where is depicted distribution of dimensionless concentration in dependence on dimensionless spacial coordinate ξ for the found pair of parameters ϱ and γ. The parameter of curves in the figure is the time interval t elapsed from the moment of addition of the indicating compound.

Physical significance of parameters of the model can be considered on basis of Eqs (*13*) abd (*14*), or from assumptions which determine coefficients of these equations. From these considerations and also from considerations made earlier[3] results that the parameter α represents the coefficient of laminar friction and should be first of all a function of fluid properties. Rotational speed should not affect this quantity. Experimental results do not exactly confirm this conclusion, but in the region of turbulent flow there does not exist a significant trend between rotational speed of impeller and this parameter. As has been already stated, the mean value of this parameter has been calculated for one geometrical arrangement. From Fig. 4 is obvious that the difference between the optimal value of parameter α (calculated by the method of nonlinear regression) and average value has a little effect on calculated dependence of concentration on time. (The minimum sum of square deviations in the phase space of parameters is perhaps flat, with respect to α).

From Eqs (*14*) or (*15*) is obvious that the parameter β is the mean velocity of convective stream of indicating particles and thus also of liquid velocity which is carrying these particles. It is possible to expect that with increasing rotational speed of impeller this velocity will increase. From the second equation (*29*) there results that also the dimensionless velocity ϱ increases with increasing speed of mixer rotation.

As has been found earlier[1,3], the parameter ε^2 in Eq. (*14*) or the dimensionless parameter γ, defined by the third of relations (*25*) can be considered to be the dimensional or dimensionless turbulent diffusivity. As intensity of turbulence increase with increasing speed of rotation it is possible to expect also the increase in value of parameter γ with this operating condition. As is obvious from Fig. 3 both these conclusions were experimentally verified, while the rise in parameters ϱ and γ is roughly linear with the rotational speed of impeller.

As concerns the boundary conditions (7) *for solution of Eqs* (16) it is necessary to mention that physical sense of these boundary condition is more illustrative and simpler than in the usual case of "unidimensional" diffusion described in general by relation[5]

$$\frac{\partial \varphi(x, t)}{\partial t} + \frac{\partial u(x, t)\, \varphi(x; t)}{\partial x} - \frac{1}{2} \frac{\partial^2 \sigma^2(x, t)\, \varphi(x; t)}{\partial x^2} = 0\,, \tag{30}$$

in which $\varphi(x; t)$ denotes probability density of particle location, $u(x, t)$ drift velocity and $\sigma^2(x, t)$ diffusion coefficient. Boundary conditions for reflecting boundary of the interval always include the relation $\partial\sigma^2(x, t)\, \varphi(z, t)/\partial x$; while at explanation of their physical sense it is necessary to admit that particle hits the boundary with infinitely large velocity. Elastic reflection of indicating particle, described by Eqs (*11*) and (*12*), enables to consider finite velocities and the corresponding distribution of velocities is then described by condition (*7*). It is also worth mentioning that Eq. (*21*) can be also written with regard to the remark made next to Eq. (*20*), *i.e.* that there exist (also with regard to Eq. (*8*)) two relations

$$\varphi_n^+(x; t) \equiv \mathsf{E}\left[\int_{-\infty}^{+\infty} f_n^+(x, v; t)\, \mathrm{d}v\right] \tag{31}$$

and

$$\varphi_n^-(x; t) \equiv \mathsf{E}\left[\int_{-\infty}^{+\infty} f_n^-(x, v; t)\, \mathrm{d}v\right]$$

which are solutions of the "unidimensional" differential equation with diffusion coefficient which is a function of time

$$\frac{\partial \varphi_n^\pm(x; t)}{\partial t} \pm \beta \frac{\partial \varphi_n^\pm(x; t)}{\partial x} - \frac{\varepsilon^2}{\alpha}\left(1 - \exp\left(-\alpha t\right)\right) \frac{\partial^2 \varphi_n^\pm(x; t)}{\partial x^2} = 0\,, \tag{32}$$

(compare Eq. (*34*) in the previous paper[3]).

The corresponding boundary conditions are given by relations

$$\sum_{n=-\infty}^{+\infty} \left[\beta - \frac{\varepsilon^2}{\alpha}\left(1 - \exp\left(-\alpha t\right)\right) \frac{\partial}{\partial x}\right] \varphi_n^+ =$$
$$= - \sum_{n=-\infty}^{+\infty} \left[-\beta - \frac{\varepsilon^2}{\alpha}\left(1 - \exp\left(-\alpha t\right)\right) \frac{\partial}{\partial x}\right] \varphi_n^-\,, \tag{33}$$

for $x \uparrow 0$ or $x \downarrow L$, which also describe reflection of the indicating particle in point, $x = 0$ and $x = L$.

This approach is thus suitable for description of diffusion processes in a limited interval.

APPENDIX

Solution of Kolmogorov Equations

It has been proved[9] that the fundamental solution of Kolmogorov diffusion equations is a function proportional to the probability density of the multiple distribution at the assumption that coefficients with first derivatives are linear functions of "spatial" variables and coefficients with second derivatives are constants. Parame'ers of this function are then function of time and they can be found by solution of the system of ordinary differential equations. In the case of Eq. (*16*) they hold for expected values of variables x and v with dots above letters for time derivatives

$$\dot{x}^{\pm} - v^{\pm} = 0$$
$$\dot{v}^{\pm} + \alpha v^{\pm} = \pm\alpha\beta\,. \qquad (34)$$

Initial conditions to these relations are determined by Eqs (*15*) and (*17*)

$$x^{\pm}(0) = 0$$
$$v^{\pm}(0) = \pm\beta\,. \qquad (35)$$

Solution of these equations are obviously relations

$$x^{\pm} = \pm\beta t$$
$$v^{\pm} = \pm\beta\,. \qquad (36)$$

For second moments of variables x and v it is possible to write this system as

$$\dot{h}_{xx} - 2h_{xv} = 0$$
$$\dot{h}_{xv} + \alpha h_{xv} - h_{vv} = 0$$
$$\dot{h}_{vv} + 2\alpha h_{vv} = 2\alpha\varepsilon^2\,. \qquad (37)$$

Initial conditions are again determined by relations (*16*) and (*17*)

$$h_{xx}(0) = h_{xv}(0) = 0\,; \quad h_{vv} = \varepsilon^2\,. \qquad (38)$$

By solving this system the last three terms of the system (*19*) are obtained.

For particular solution $f_n^{\pm}$ it is necessary to modify the first initial condition (*35*) so that the expected position of the particle in moments $t = 2nL/\beta$ is equal to zero and in moments $(2n + 1)\,L/\beta$ is equal to L, here n is the integer.

The proposed procedure can be applied for stationary solution of Eqs (*14*); in last relations (*34*) and (*37*) zero time derivatives are considered so that relations are obtained $v^{\pm} = \pm\beta$; $h_{vv} = \varepsilon^2$ which are parameters of Eq. (*15*).

The authors like to thank Miss J. Poškubková for carefully performed computations.

LIST OF SYMBOLS

b coefficient in Eq. (*13*) ms^{-2}
c coefficient in Eq. (*13*) $ms^{-3/2}$
f probability density
g function characterising nonrandom force ms^{-2}
h second central moment
h function characterising random force $ms^{-3/2}$
n summation index
r correlation coefficient
t time s
u drift velocity ms^{-1}
v velocity ms^{-1}
x coordinate of the point m
E operator of expected value
L length of mixer m
Q concentration of indicating particles kgm^{-3}
S^2 variance of dimensionless concentration
V random particle velocity ms^{-1}
Var variance operator
W Wiener process $s^{1/2}$
X random location of particle m
Z dimensionless concentration
α coefficient in Eq. (*13*) s^{-1}
β parameter of Eq. (*14*), mean velocity of indicating particles ms^{-1}
γ parameter of the model defined by Eq. (*29*)
δ Dirac function
ε parameter of Eq. (*14*), variance of velocities of indicating particles ms^{-1}
φ probability density
ξ dimensionless linear coordinate
ϱ parameter of model Eq. (*29*)
σ^2 diffusion coefficient $m^2 s^{-1}$
τ time s
θ dimensionless time

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Translated by M. Rylek.